Evaluation of Multiparametric Linear Programming for Economic Dispatch under Uncertainty

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***Abstract*—For risk assessment purposes, we study how economic dispatch decisions vary with the uncertain input factors that may arise, e.g., from the use of variable renewable energies. Given a known random input distribution and linear programming (LP)-based dispatch, we aim to describe the distribution of the resulting variables and objective values. Relying on Monte Carlo simulation (MCS) is computationally expensive, especially if the uncertain factors are high dimensional. In this paper we evaluate an algorithm using multiparametric linear programming (MPLP) for this purpose. It avoids solving an LP for every sample of the random vector by characterizing the parametric LP solution as a piece-wise linear function whose pieces can be stored for repeated use. We compare the algorithm with MCS and other quasi-Monte Carlo sampling approaches for three economic dispatch use cases with varying complexity. The MPLP approach is as accurate as MCS, but up to 300 times faster for the merit order use case.**

***Index Terms*—Linear programming, Uncertain systems, Sampling methods, Power system simulation.**

I. Introduction

An increase in renewable energy generation has been observed in the last years, providing cleaner and sometimes even cheaper energy to consumers [1]. A serious drawback of systems with a large share of wind and solar energy is that their power output is highly variable and not fully predictable due to ever-changing weather conditions. The outcomes of dispatch routines depending on these conditions are then also uncertain. This presents a challenge to system operators who require at least a description of the probabilities of potential variations to ensure safe system operation.

Linear programming (LP) is a common tool for solving economic dispatch in energy systems [2], [3]. Monte Carlo Simulation (MCS) can be used to determine the distribution of the dispatch outcomes given a known distribution of the input parameters. With a large amount of samples, it is known to produce accurate results. Still, this brute-force approach requires many computational resources to solve medium and large problems [4].

In this work, we evaluate a multiparametric linear programming (MPLP) approach to solve LPs under uncertainty orders of magnitude faster than MCS. It can solve problems with uncertainty in the right-hand side of the constraints and in the objective function. We use three economic dispatch use cases with varying complexity to validate the algorithm and to compare it with three sampling-based methods: MCS, Latin Hypercube sampling (LHS) [5], and Halton Sequence sampling (HSS) [6].

The remainder of the paper is structured as follows: in Section II we describe the problem statement. We present the MPLP approach to solve LPs under uncertainty in Section III. In Section IV we define the three economic dispatch use cases with which we evaluate the algorithm. Section V presents the results of our evaluation, and in Section VI we provide an outlook and discussion on future research.

A. Related Work

Uncertainty in LPs has long been investigated in diverse fields of study. For a broader literature review on the topic, refer, e.g., to [7]. In [8], a fast method to solve LPs under uncertainties is proposed. The randomness is assumed Gaussian distributed with low variance, such that the optimal basis of the problem remains unchanged. Pure MCS is used in [9], [10] to solve LPs under uncertainty. In [9], MCS is used to model the uncertainties related to the stochastic variations of wind power generation and load demand. Other works propose the combination of different techniques to improve the speed of MCS. In [11], a new probabilistic method involving transient stability and voltage stability for power system security assessment is presented by combining MCS and a neural network. The work in [12] offers potential improvement by using quasi-Monte Carlo (QMC) techniques, enabling faster convergence of the simulation.

One can achieve considerable speed improvements compared to the above described sampling-based approaches by leveraging MPLP. A forecasting technique of real-time locational marginal cost using MPLP was developed in [13]. In [14], a fast reliability evaluation of power systems is proposed, where the generation status and the sampled load are the MPLP parameters of the model. The algorithm evaluated in our work is analogous to the one proposed in [14], with an extension for LPs with uncertainties in the objective function.

II. Problem Statement

A parametric LP with variable right-hand side (RHS) of the equality constraints is given as

$$\begin{aligned} \mathbf{x}^*(\boldsymbol{\theta}) \in \arg\min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} + \mathbf{B}_\theta\boldsymbol{\theta}, \\ & \mathbf{x} \geq 0, \end{aligned} \tag{1}$$

Algorithm 1 MPLP-based Algorithm

Input: Number of samples N

Output: Optimal solutions $\tilde{\mathbf{x}}$, Optimal costs $\tilde{z}$

```
x̃, z̃ := ∅
Draw N samples of θ and store them in θ̃.
while θ̃ ≠ ∅ do
    Solve LP with first element of θ̃ and calculate CR Θ with (3) or (5).
    With the samples of θ̃ that belong to Θ, calculate x* and z* and append them to x̃, z̃.
    Remove utilized samples from θ̃.
end while
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where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{B}_\theta \in \mathbb{R}^{m \times p}$, and $\mathbf{c} \in \mathbb{R}^n$ are parameters of the model and $\boldsymbol{\theta} \in \mathbb{R}^p$ is a random parameter. In case the variability is in the objective function (OFC), the parametric linear program is given as

$$\begin{aligned} \mathbf{x}^*(\boldsymbol{\theta}) \in \arg\min_{\mathbf{x}} \quad & (\mathbf{c} + \mathbf{C}_\theta \boldsymbol{\theta})^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq 0, \end{aligned} \tag{2}$$

with $\mathbf{C}_\theta \in \mathbb{R}^{n \times p}$ being another parameter of the model.

Assuming that $\boldsymbol{\theta}$ is a random vector drawn from a distribution with probability density function (PDF) $p_\Theta(\boldsymbol{\theta})$, the solution $\mathbf{x}^*(\boldsymbol{\theta})$ of (1) or (2) is also a random vector and we denote its PDF by $p_{X^*|\Theta}(\mathbf{x}^*|\boldsymbol{\theta})$.

In many risk assessment problems, $p_\Theta(\boldsymbol{\theta})$ is known. The PDF of the solution $p_{X^*|\Theta}(\mathbf{x}^*|\boldsymbol{\theta})$ can, however, not be derived from it in closed form, at least not for general LPs. This holds even if the input distribution is the standard multivariate Gaussian distribution $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, which we will assume throughout this paper.

One way to solve this task is to obtain a discrete approximation of $p_{X^*|\Theta}(\mathbf{x}^*|\boldsymbol{\theta})$ using samples of $\boldsymbol{\theta} \sim p_\Theta(\boldsymbol{\theta})$. These can be drawn with low computational effort [15], but the method requires the solving of the LP for each sample point, making this approach computational intensive.

In this paper, we evaluate an algorithm that proposes a more efficient way of obtaining a discrete approximation of $p_{X^*|\Theta}(\mathbf{x}^*|\boldsymbol{\theta})$ by leveraging MPLP to eliminate unnecessary calls of the LP optimization algorithm.

III. Multiparametric Linear Programming Algorithm

A. Multiparametric Linear Programming

Following [16] every solution (if existent) of LP (1) or (2) defines an optimal basis. For a subset of m basic variables with indices $\mathbf{I}_B$, matrix $\mathbf{B} \in \mathbb{R}^{m \times m}$ consisting of m columns of $\mathbf{A}$ indexed by $\mathbf{I}_B$ is called a basis, and $\mathbf{D} \in \mathbb{R}^{m \times (n-m)}$ denotes the matrix with the remaining columns of $\mathbf{A}$. In the case $\boldsymbol{\theta} = 0$, if $\mathbf{B}$ is non-singular, we can define the relative cost vector for the non-basic variables as $\mathbf{r}_D^T = \mathbf{c}_D^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{D}$. Here, $\mathbf{c}_B$ is a vector with m components of $\mathbf{c}$ indexed by $\mathbf{I}_B$, and $\mathbf{c}_D$ contains the remaining components of $\mathbf{c}$. Vector $\mathbf{x} = (\mathbf{x}_B, 0)$, with $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, is an optimal solution of (1) or (2) iff $\mathbf{r}_D$ and $\mathbf{x}_B$ are both non-negative.

While primal feasibility in (1) depends on the random vector $\boldsymbol{\theta}$, $\mathbf{r}_D$ does not change with $\boldsymbol{\theta}$. Thus, basis $\mathbf{B}$ remains optimal as long as

$$-\mathbf{B}^{-1}\mathbf{B}_\theta \boldsymbol{\theta} \leq \mathbf{B}^{-1}\mathbf{b}. \tag{3}$$

The polyhedron $\Theta_{RHS}(\mathbf{B}) = \{\boldsymbol{\theta} | \text{ (3)}\}$ is called the critical region (CR) where $\mathbf{B}$ is an optimal basis, and, for any $\boldsymbol{\theta} \in \Theta_{RHS}(\mathbf{B})$, an optimal solution of (1) is given by:

$$\mathbf{x}^* = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{B}_\theta \boldsymbol{\theta}. \tag{4}$$

In the OFC problem of (2), only $\mathbf{r}_D$ depends on $\boldsymbol{\theta}$. Thus, basis $\mathbf{B}$ remains optimal if

$$\boldsymbol{\theta}^T \left(-\mathbf{C}_{\theta,D}^T + \mathbf{C}_{\theta,B}^T \mathbf{B}^{-1}\mathbf{D} \right) \leq \mathbf{c}_D^T - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{D}, \tag{5}$$

with $\mathbf{C}_{\theta,B}$ being a matrix with m rows of $\mathbf{C}_\theta$ indexed by $\mathbf{I}_B$, and $\mathbf{C}_{\theta,D}$ being the matrix with the remaining rows. The critical regions are given by $\Theta_{OFC}(\mathbf{B}) = \{\boldsymbol{\theta} | \text{ (5)}\}$, and the value of the objective function for $\boldsymbol{\theta} \in \Theta_{OFC}(\mathbf{B})$ is given by:

$$\mathbf{z}^* = \left(\mathbf{c}_B + \mathbf{C}_{\theta,B}\boldsymbol{\theta}\right)^T \mathbf{x}_B. \tag{6}$$

B. MPLP-based Algorithm to solve LP under Uncertainty

Here, we present an MPLP-based algorithm to efficiently solve LPs with either RHS or OFC uncertainties. As mentioned earlier, the algorithm is reminiscent of [14]. The key idea of it is to use already calculated CRs to quickly evaluate the optimal value of the decision variable $\mathbf{x}$ and of the total cost (objective function) z. The algorithm is formulated in Algorithm 1 and is summarized in the following.

An amount of N samples of the random vector $\boldsymbol{\theta}$ are drawn and stored in an array. Sampling can be performed using any method, e.g. MCS or LHS. The first sample is used to solve the LP and, given the resulting optimal basis, the CR is calculated using (3) in case of an RHS problem, or (5) if it is an OFC problem. Then, the algorithm calculates and stores an optimal value of the decision variables and of the total cost for all samples within this CR. Samples within the CR are removed from the samples array, and the process is repeated until the array is empty.

As observed in Section V, this algorithm is particularly efficient when the LP's solution has a small amount of large CRs. However, when the solution has many small CRs, the algorithm performs no better (or even worse) than pure sampling methods like MCS. These points are discussed in Section V.

IV. Evaluated Use Cases

We examine three use cases for the economic dispatch to evaluate the performance of the MPLP-based algorithm. Samples of random vector $\boldsymbol{\theta}$ are supposed to follow a multivariate Gaussian distribution. The values of the problems' parameters can be found in the Appendix.

A. *Merit Order with Uncertain Demand (MO)*

Ten power plants have to supply energy to an uncertain load with mean d. Transmission losses are neglected. The power plants have different marginal costs $\mathbf{c}$ and generation upper limits $\bar{\mathbf{x}}$. The LP of the use case is the following:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{1}^T\mathbf{x} = d + \theta, \\ & 0 \leq \mathbf{x} \leq \bar{\mathbf{x}}, \end{aligned} \tag{7}$$

where $\mathbf{1}$ is a vector of ones with appropriate dimension. The solution of this problem (calculated using MCS with a large amount of samples) has ten CRs, each one corresponding to a generation upper limit.

B. *Bidding Strategy for Thermal Power Plant with Ramping Constraints (BS)*

In the day-ahead planning of a thermal power plant, the goal is to maximize the profit by selling the production to the energy market. We consider a combined cycle gas power plant that has a ramping constraint δ and a maximal generation capacity $\bar{x}$. Profit is given by the random energy price $\mathbf{p}_e + \boldsymbol{\theta}$ that varies throughout the planning horizon minus the plant's fixed marginal cost c_m. Fig. 1 shows the price curve that was utilized in this use case. The mean prices are taken from the Entso-E transparency site [17] for the days 21–22 of October in the German-Luxemburg market zone. The NetConnect gas price for October 21 was near 15€/MWh [18], and the emission allowances was 24€/tCO2 [19]. The specific carbon dioxide emissions for natural gas is assumed to be 0.2 tCO2/MWh [20]. The marginal cost c_m is calculated assuming a total power plant efficiency of 0.56.

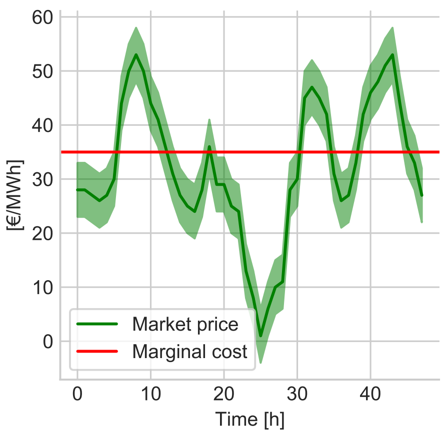

Fig. 1. Market price and marginal cost of the thermal power plant for the BS use case. The green line represents the mean market price and the shaded area plus minus one standard deviation.

Considering a period of 48 hours, the LP for the use case is:

$$\begin{aligned} \min_{\mathbf{x}} \quad & (c_m\mathbf{1} - \mathbf{p}_e - \boldsymbol{\theta})^T\mathbf{x} \\ \text{s.t.} \quad & |x_t - x_{t-1}| \leq \delta, \ \forall t \in [2, 48], \\ & 0 \leq \mathbf{x} \leq \bar{\mathbf{x}}. \end{aligned} \tag{8}$$

This problem is much harder to solve than the MO's one, with its solution having more than 8000 CRs.

C. *Battery Management for Solar-powered Microgrids (SM)*

The last use case is a microgrid management problem for renewable islands where we aim at minimizing the production cost given uncertainty in the renewable generation. The micro-grid has three components: a diesel generator with maximal power output $\bar{x}_{\mathcal{D}}$, a photovoltaic plant with uncertain, time-dependent maximal output $\bar{x}_{P,t} + \theta_t$, shown in Fig. 2, a battery (capacity $\bar{x}^e_\beta$, maximal power output $\bar{x}^p_\beta$, self-discharge rate λ, efficiency η), and a constant load d. The only generation cost in the system is the diesel fuel needed to run the diesel generator with specific cost c. The LP of the use case is as follows:

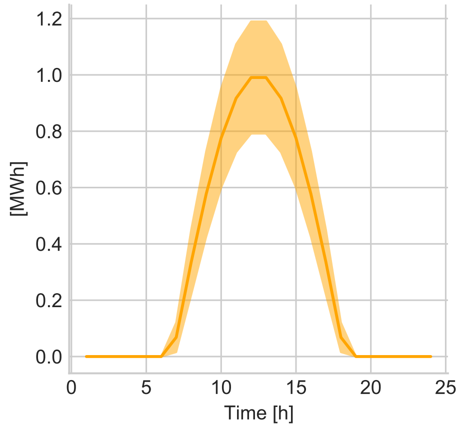

Fig. 2. Time-dependent maximum power output of photovoltaic plant for the SM use case. The yellow line represents the mean maximum power output and the shaded area plus minus one standard deviation.

$$\begin{aligned} \min_{\mathbf{x}} \quad & c\mathbf{1}^T\mathbf{x}_{\mathcal{D}} \\ \text{s.t.} \quad & \mathbf{x}_{\mathcal{D}} + \mathbf{x}_P + \mathbf{x}^+_\beta - \mathbf{x}^-_\beta = d\mathbf{1}, \\ & 0 \leq \mathbf{x}_{\mathcal{D}} \leqslant \bar{x}_{\mathcal{D}}\mathbf{1}, \\ & 0 \leq \mathbf{x}_P \leqslant \bar{\mathbf{x}}_P + \boldsymbol{\theta}, \\ & 0 \leq \mathbf{x}^e_\beta \leqslant \bar{x}^e_\beta\mathbf{1}, \\ & 0 \leq \mathbf{x}^+_\beta, \mathbf{x}^-_\beta \\ & |x^+_{\beta,t} - x^-_{\beta,t}| \leqslant \bar{x}^p_\beta, \ \forall t \in [2, 24] \\ & x^e_{\beta,t} - \lambda x^e_{\beta,t-1} = -\frac{1}{\eta}x^+_{\beta,t} + \eta x^-_{\beta,t}, \ \forall t \in [2, 24] \end{aligned} \tag{9}$$

where $\mathbf{x} = (\mathbf{x}_{\mathcal{D}}, \mathbf{x}_P, \mathbf{x}^+_\beta, \mathbf{x}^-_\beta, \mathbf{x}^e_\beta)$ are the power production of the diesel generator, the power production of the photovoltaic

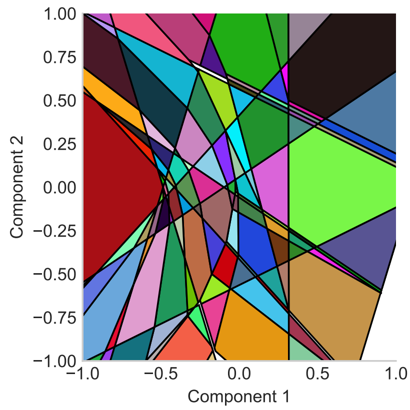

Fig. 3. Critical regions of SM use case's solution as function of the first two principal components of $\boldsymbol{\theta}$.

plant, the output/input power provided/consumed by the battery, and the battery's energy level for each of the 24 hours of the day. The solution of this problem has approximately 800 CRs. In Fig. 3 we show how the CRs are distributed in the $\mathbb{R}^2$ plane spanned by the first two eigenvectors of a Principal Component Analysis [21] of $\boldsymbol{\theta}$.

V. Results

A. Evaluation Criterion

The accuracy of the MPLP-based algorithm is evaluated with a metric analogous to the Cramér-von Mises criterion [22], [23], where the reference distribution is the solution obtained with MCS using a large set of samples, referred to as *benchmark*. In our use cases, it was verified experimentally that the benchmark's distribution was stable w.r.t. sampling variations given 10^4 samples. For each method K, which can be either the MPLP-based algorithm or a sampling approach, the accuracy metric is defined by the root-mean-square error (RMSE) over 99 evenly-distributed quantiles of the inverse cumulative distribution function of the optimal cost obtained by method K, i.e.,

$$Acc_K = \sqrt{\frac{\sum_{i=1}^{99} \left[\mathbf{F}_K^{-1}(i/100) - \mathbf{F}_{Bench}^{-1}(i/100)\right]^2}{99}}, \quad (10)$$

where $\mathbf{F}_K^{-1}$ and $\mathbf{F}_{Bench}^{-1}$ are the inverse cumulative distribution functions of the optimal cost obtained by method K and by the benchmark, respectively.

B. Methods Comparison

Five different methods to solve LPs under uncertainty are compared regarding accuracy and solving speed: three sampling methods, MCS, LHS and HSS, and two variations of the MPLP-based algorithm, one using MCS and another using LHS to sample from the random vector $\boldsymbol{\theta}$. The results can be seen in Fig. 4. They were obtained on an Intel Core i5-3210M @ 2.50GHz CPU laptop with 16GB using Gurobi [24] as the LP solver and with Python as the modeling programming language.

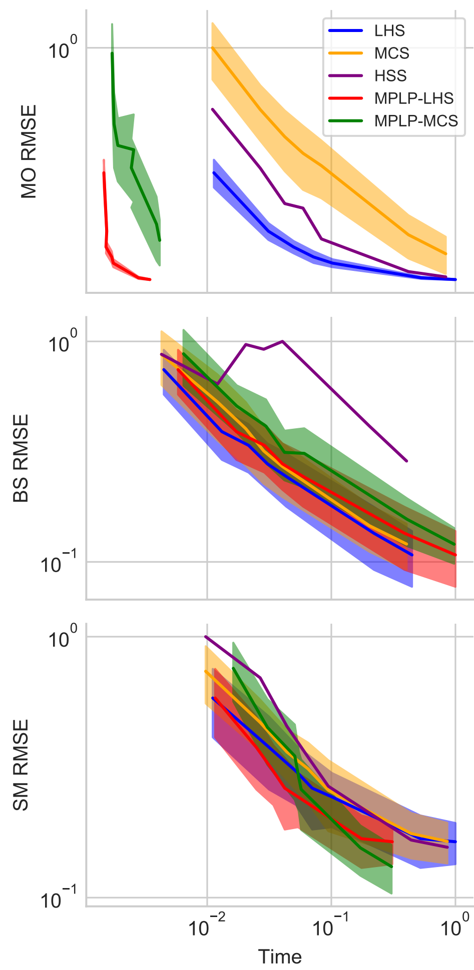

Fig. 4. Accuracy (10) of the optimal cost distribution as a function of computation time. Values from both dimensions were scaled with min-max normalization (using the average values over all runs). As above, the line represents mean values and the shaded areas plus minus one standard deviation over 50 different sample sets.

For the cases with a smaller number of CRs, a clear advantage with the MPLP-based method is observed. For instance, the MPLP-based algorithm solves the MO problem two orders of magnitude faster than the sampling methods, with the variant using LHS having a 291-fold speedup. In the SM use case, the difference in solving speeds is around 3.2 times between the MPLP-based algorithm and the sampling approaches. The MPLP-based algorithm performs worse than the sampling methods in the BS use case because when the solution has a large amount of CRs calculating the CRs and verifying which samples are within the CRs is more time consuming than just solving the individual LPs. Regarding the sampling methods, LHS is faster than MCS and HSS in all three use cases.

VI. Conclusion and Future Work

This paper presents an evaluation of an MPLP-based algorithm to solve economic dispatch LPs under uncertainty.

Different methods for sampling distributions are compared regarding accuracy and computation time. It was verified experimentally that the performance of the MPLP-based algorithm depends on the structure of the LP. If the solution has only a few CRs, and, thus, a high number of samples per base, the MPLP-based algorithm is up to 291 times faster than MCS with the same accuracy. Conversely, if the solution has a large amount of CRs (for instance, in the bidding strategy use case), sampling methods are faster, with LHS being the best overall variant.

Taking into account this work's results, future research can be done in exploring common patterns of economic dispatch LPs, so that it is possible to choose for each problem the method with the best tradeoff between computation time and accuracy. Furthermore, it would be interesting to investigate the possibility of a generalized LP relaxation framework, with which the number of CRs in the solution is reduced, giving a computational advantage to the MPLP-based algorithm.

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APPENDIX

TABLE I
MO PARAMETERS

$\mathbf{c}$	$[39, 40, 71, 79, 51, 55, 70, 41, 37, 34]$ €/MWh
$\bar{\mathbf{x}}$	$[170, 150, 160, 270, 90, 120, 80, 60, 210, 110]$ MWh
d	710 MWh
μ	0 MWh
Σ	$\frac{d^2}{25}$ MWh^2

TABLE II
BS PARAMETERS

$\bar{x}$	500 MWh
Δ	300 MWh/h
c_m	35.36 €/MWh
$\boldsymbol{\mu}$	0 MWh
Σ_{ij}	$25e^{\frac{-(i-j)^2}{8}}$ MWh^2

TABLE III
SM PARAMETERS

c	325 €/MWh
d	0.75 MW
$\bar{x}_{P,13}$	1 MW
$\bar{x}_{\mathcal{D}}$	1 MW
$\bar{x}_{\beta}^{p}$	0.75 MW
λ	0.99
η	0.95
$\bar{x}_{\beta}^{e}$	0.75 MWh
$\boldsymbol{\mu}$	0 MW
Σ_{ij}	$0.04\frac{sin(\frac{\pi \cdot i}{12})+sin(\frac{\pi \cdot j}{12})}{2}e^{\frac{-(i-j)^2}{4}}$ MW^2